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On the spatial stability of free-convection flows in a saturated porous medium

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Abstract

We consider the free-convection boundary-layer flow in a saturated porous medium adjacent to an impermeable vertical surface. It is assumed that the surface is supplying heat to the porous medium in a prescribed way, which varies along the surface. The problem, which relates to the spatial stability of the known similarity solutions of the boundary-layer equations, is formulated and certain analytical results presented for special cases. For this special class of flows we are able to determine analytically the first eigenvalue for all relevant parameter values and thereby show that such flows are spatially stable.

1. Introduction

The flow due to steady vertical convection from a heat source in a saturated porous medium has motivated a number of studies because of its possible geothermal applications. Of particular interest is the free-convection flow over a vertical flat plate adjacent to such a saturated porous medium when there is (i) a prescribed temperature or (ii) a heat flux distribution at the surface of the plate. In the study by Cheng and Minkowycz [1] the boundary-layer hypothesis is invoked and similarity solutions obtained for case (i) where the wall temperature varies as a power of the distance along the plate. Merkin [2], again using a boundary-layer formulation, has considered the problem relating to (ii) but with constant heat flux and the addition of a uniform stream. Amongst other things he was able to show that when the stream and buoyancy force are in the same direction the leading term in the solution far downstream is of the constant heat-flux free-convection type. Merkin [3] later gave numerical solutions to the similarity equations corresponding to the free convection driven by a heat flux which varied as the power of the distance along the plate.

The purpose of the present paper is to consider the physical reality of such flows as those discussed by Merkin [3] by investigating their spatial stability. The need for a knowledge of the resulting eigenvalues is evidenced by a remark in Daniels and Simpkins [4], albeit concerning a different problem, that such information was unknown.

In what follows we are able to obtain analytically the totality of the eigenvalues for one parameter value and also to give the first eigenfunction. Further analytical results are given and we note an unusual property of the eigenvalue problem: it is possible to obtain the first eigenvalue analytically for all the relevant parameter values which define the primary flow. The analysis leading to the first eigenvalues involves simple consequences of the governing equation plus the basic property of the first eigenfunction.

2. The governing equation and analysis

The problem to be considered here concerns the free-convection flow over a vertical flat plate that supplies heat to a saturated porous medium in which it is embedded. It is assumed that the convected fluid and the porous medium are in thermodynamic equilibrium, that they have uniform physical properties and that we can invoke the Boussinesq approximation. It is further assumed that the equation of motion is replaced by Darcy's law and that the Rayleigh number is sufficiently large so that the simplifying boundary-layer hypothesis can be introduced. If we choose a cartesian coordinate system $0xy$ so that $y=0$ is the plate and x is measured in the direction of the buoyancy force, the boundary-layer equations which govern the flow are

$$u=(g\beta K\rho_\infty/\mu)(T-T_\infty), \tag{1}$$

$$u_x+v_y=0, \tag{2}$$

$$uT_x+vT_y=\alpha T_{yy} \tag{3}$$

where u, v are the flow velocity components in the directions of x and y increasing respectively, T is the temperature, ρ is the density and β, K, μ, α are respectively the coefficient of thermal expansion, the modified permeability of the saturated porous medium, the viscosity and the thermal diffusivity. Further, g represents the acceleration due to gravity and the suffix ∞ indicates conditions in the ambient fluid far away from the plate. The situations we are interested in here give rise to the boundary conditions

$$\begin{aligned} &v=0, \quad \partial T/\partial y=-q(x)/k \quad \text{at } y=0,\\ &u\to 0, \quad T\to T_\infty \quad \text{as } y\to\infty, \end{aligned} \tag{4}$$

where $q(x)$ is the local heat-transfer rate at the plate and k is the thermal conductivity of the porous medium. In what follows it will be convenient to write $q(x)=qt(x)$ where $t(x)$ represents the (non-dimensional) variation of heat flux.

In terms of the stream function $\psi(x, y)$, where $u=\psi_y$, $v=-\psi_x$, we non-dimensionalise and transform the variables by writing

$$\psi=A\xi f(\xi, \eta), \tag{5}$$

$$T=T_\infty[1+\theta(\xi, \eta)], \tag{6}$$

where

$$A=g\beta K\rho_\infty kT_\infty^2/(\mu q),\ \xi=\left[\tfrac{3}{2}\int^X[t(X)]^{1/2}\,\mathrm{d}X\right]^{2/3},\quad X=(\alpha\mu q^2)x/\left(g\beta K\rho_\infty k^2T_\infty^3\right)$$

and $\eta=t^{1/2}qy/\left(kT_\infty\xi^{1/2}\right)$.

Equation (2) is, of course, automatically satisfied by the introduction of the stream function, and equation (1) implies that $\theta=(t\xi)^{1/2}f_\eta$. Substitution of the latter into (3) leads to the single equation

$$f_{\eta\eta\eta}+ff_{\eta\eta}-B(\xi)f_\eta^2=\xi(f_\eta f_{\eta\xi}-f_\xi f_{\eta\eta}) \tag{7}$$

where $B(\xi)=\frac{1}{2}+[3t'/(4t^{3/2})]\int^X t^{1/2}\,\mathrm{d}X$, $t'=\mathrm{d}t/\mathrm{d}x$, and the boundary conditions are

$$f=0, \quad f_{\eta\eta}=-1, \quad \text{on } \eta=0, \quad \text{and } f_\eta\to 0 \quad \text{as } \eta\to\infty. \tag{8}$$

It will be opportune at this stage to integrate equation (7) with respect to η between $\eta=0$ and infinity to obtain

$$(1+B)\int_0^{\infty} f_\eta^2 \, d\eta + \xi \frac{d}{d\xi}\int_0^{\infty} f_\eta^2 \, d\eta = 1, \tag{9}$$

providing that

$$\left[(f-\xi f_\xi) f_\eta\right]_0^{\infty} = 0. \tag{10}$$

With $t(x)=X^m$ similarity solutions are possible by taking $f(\xi, \eta)=F(\eta)$. In these circumstances $B(\xi)=$ constant, β, where $\beta=(1+2m)/(2+m)$ and the function $F(\eta)$ satisfies the equation

$$F''' + FF'' - \beta F'^2 = 0 \tag{11}$$

and the boundary conditions are

$$F(0)=0, \quad F''(0)=-1, \quad \text{and } F'(\eta)\to 0 \quad \text{as } \eta\to\infty. \tag{12}$$

The similarity solution defined by (11) and (12) corresponding to $\beta=\frac{1}{2}$ arose in the work of Merkin [2] who later, [3], gave solutions for a range of values of β. We note here that the solutions discussed by Merkin are such that the condition in (11) is satisfied.

The purpose of this paper is to present some results concerning the spatial stability of the similarity solutions. The examination of the stability of these solutions is of importance because of the possible practical applications of such processes. We therefore perturb about the similarity solution, defined by (11) and (12), by writing

$$f(\xi, \eta) = F(\eta) + f_1(\xi, \eta). \tag{13}$$

After substituting into equation (7) with $B=\beta$ and then linearising, we find that the resulting equation for f_1 has a separable form of solution with $f_1=\xi^{-\lambda}H(\eta)$ provided that

$$H''' + FH'' + (\lambda-2\beta)F'H' + (1-\lambda)F''H = 0 \tag{14}$$

with

$$H(0)=H''(0)=0, \quad \text{and } H'(\eta)\to 0 \quad \text{as } \eta\to\infty. \tag{15}$$

Equation (14) with boundary conditions (15) constitutes an eigenvalue problem and, for all values of β for which the solution of (11) subject to (12) is known, we anticipate the eigensolutions to be complete. For the case when $\beta=1$ there is an analytic solution for $F(\eta)$, $F(\eta)=1-e^{-\eta}$, and it is possible to obtain analytic results for the eigenvalue problem.

For $\beta=1$ we proceed in a manner similar to that of Banks and Zaturska [5] where the eigenvalues corresponding to the flow arising due to a stretching wall were investigated. We put $\phi(\eta)=(H/F')'$ and find that with $\phi(\eta)=\Phi(s)/s$ where $s=-e^{-\eta}$, the equation for Φ becomes

$$s\Phi'' + (1-s)\Phi' - (\lambda-1)\Phi = 0. \tag{16}$$

This is the confluent hypergeometric equation with solutions

$$\Phi_1 = {}_1F_1(\lambda-1, 1, s)$$

and

$$\Phi_2 = \Phi_1(s)\log(s) + P(s)$$

where $P(s)$ is a power series in s. The general solution of (14) can therefore be written

$$H(\eta) = B(1-F) + C(1-F)\int_0^F \frac{\Phi_1(F-1)}{(1-F)^2}\,\mathrm{d}F$$
$$+ D(1-F)\int_0^F \left[\frac{\Phi_1(F-1)\log(F-1) + P(F-1)}{(1-F)^2}\right]\mathrm{d}F \tag{17}$$

where B, C, D are constants of integration and $F = 1 - e^{-\eta}$.

To satisfy the last condition in (15) necessitates $D = 0$, and we take $B = 0$ to satisfy $H(0) = 0$. The complete set of eigenvalues is then found by imposing $H''(0) = 0$; on carrying out the details we obtain

$$_1F_1'(\lambda - 1, 1, -1) = {}_1F_1(\lambda - 1, 1, -1) \tag{18}$$

for the determination of λ. The latter relationship can be written (see Abramowitz and Stegun [6]) in the equivalent form

$$(2-\lambda)\{_1F_1(3-\lambda, 1, 1) - {}_1F_1(2-\lambda, 1, 1)\} = 0. \tag{19}$$

Hence one root is $\lambda = 2$ and the equation for the determination of the other roots can be written

$$_1F_1(3-\lambda, 2, 1) = 0. \tag{20}$$

From equation (20) it is possible to infer, from Jahnke, Emde and Lösch [7], that there are no solutions with λ negative, and reference to Slater [8] enables us to deduce that another eigenvalue is $\lambda = 5.75362$. We therefore take $\lambda = 2$ to be the first eigenvalue and $\lambda = 5.75362$ to be the second. Further, for λ large the asymptotic properties of the confluent hypergeometric function given in [6] lead to the eigenvalue relationship

$$\lambda \approx 2 + [\pi(l + 5/4)/2]^2 \tag{21}$$

for $l = -1, 0, 1, 2, \ldots$. We note here that with $l = -1$ we obtain 2.154 and with $l = 0$ $\lambda = 5.855$ which we identify as the first and second eigenvalues respectively. The first eigenfunction can also be found fairly easily: normalising the eigenfunction by imposing $H'(0) = 1$, we find that $C = e$ in (17) and that

$$H_1(\eta) = e^F + (1-F)[eE_1(1) - 1 - eE_1(1-F)], \tag{22}$$

where $F = 1 - e^{-\eta}$ and $E_1(\theta)$ is the exponential integral.

It is easy also to verify that for $\beta = \frac{1}{2}$ there is an eigensolution $\lambda = 3/2$, $H(\eta) = 2F - \eta F'$ and, from the numerically known solution of $F(\eta)$ for this value of β, we note that it is the first eigensolution. It corresponds of course to a shift in the origin along the x-axis. We might also note in passing that the function, $2F - \eta F'$, arose in an expansion carried out by Merkin [2], who identified it as a complementary function, but it is more helpful to recognise it as an eigenfunction.

There is another special case where a knowledge of the eigenfunction can be obtained analytically and it concerns $\beta = -1$. Although there is no solution of (11) which satisfies (12) for $\beta = -1$, Zaturska and Banks [9] have shown that for $0 < \epsilon \equiv 1 + \beta \ll 1$ the solution of (11) has the form

$$F(\eta) = \epsilon^{-1/3}[F_0(z) + \epsilon F_1(z) + \dots] \tag{23}$$

where $z = \eta\epsilon^{-1/3}$, $F_0(z) = a \tanh(az/2)$ and $a = 3^{1/3}$. The function $F_1(z)$ was also given in [9] but we consider here only the leading-order term. From (23) we may anticipate that the limiting form of the eigensolutions as $\beta \downarrow -1$ is given by the results corresponding to the basic flow defined by $F_0(z)$. This latter flow coincides with the well-known two-dimensional jet and we are able to deduce the necessary eigensolutions from the analysis of Riley [10] who considered the radial free jet. We find that

$$\lambda_1 = 0, \quad H_1 = (F_0 + zF_0'), \tag{24}$$

$$\lambda_2 = 3, \quad H_2 = (F_0 - 2zF_0'). \tag{25}$$

This concludes the investigation of the eigensolutions for those special values of the parameter β for which there exist analytic solutions of the similarity equation (11) which satisfy (12). However, the boundary-layer equations for this class of flows possess the interesting property that the first eigenvalue can be found directly from the governing equations for all β for which the similarity solution exists.

To obtain the first eigenvalue for arbitrary β we substitute (13) into (9), with $B(\xi) = \beta$, a constant and $f_1 = \xi^{-\lambda}H(\eta)$, and obtain

$$(1+\beta)\int_0^\infty F'^2 \, d\eta = 1 \tag{26}$$

and

$$(1+\beta-\lambda)\int_0^\infty F'H' \, d\eta = 0 \tag{27}$$

by equating terms of O(1) and $O(\xi^{-\lambda})$ respectively. These integrals can also be obtained from the equations for F and H by integrating equations (11) and (14) with respect to η between $\eta = 0$ and $\eta = \infty$. The integral constraint in (26) is equivalent to one obtained by Merkin [3] who used it to show that there is no similarity solution of the boundary-layer equations in the form $\xi F(\eta)$ for $\beta = -1$. However, the integral in (27) provides a constraint on the eigenfunction in general, although, because $F'(\eta)$ is non-negative for each β in the range $(-1, \infty)$, the integrand is of one sign if $H'(\eta)$ exists and is one-signed in the interval $0 < \eta < \infty$. We deduce from (27) therefore that the first eigenvalue is given by

$$\lambda_1 = 1 + \beta. \tag{28}$$

It should be noted that the result in (28) is consistent with the detailed analytic results presented above for the special parameter values of $\beta = 1$, 0 and the limiting value as $\beta \downarrow -1$. This method, whereby the first eigenvalue is deduced for all relevant values of β, is somewhat akin to the analysis of Banks and Zaturska [5]. In that study, which was motivated by the flow generated by a stretching wall, the first eigenvalue was also found for a range of parameter values without a detailed knowledge of the first eigenfunction.

The important implication from (28) is, of course, that the similarity flows are spatially stable for $-1 < \beta < 2$ as $x \to \infty$, and for $2 < \beta < \infty$ as $x \to 0$, precisely the regions where the respective boundary-layer flows are valid. The limiting flow as $\beta \downarrow -1$ is neutrally stable. The above analysis leading to the one-parameter family of similarity solutions fails for $\beta \leqslant -1$ (see equation (26) for example) and we have discussed the eigenvalues for $\beta = -1$ by way of a limiting procedure. However, if, for $\beta \leqslant -1$, we argue in an analogous manner to that proposed by Banks [11] we are led to the conclusion that all such flows are neutrally stable.

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